Journal of Modern Technology and Engineering
Vol.6, No.2, 2021, pp.143-150

CO-EVEN DOMINATION NUMBER OF SOME PATH RELATED GRAPHS

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Abstract. A new domination parameter co-even domination number denoted by $\gamma_{coe}(G)$ of a graph is defined by M.M. Shalaan and A.A. Omran (2020) and some properties, results are found by them on basic graph classes, path, cycle, star etc. Path related graphs and trees have many applications in real life network models. In this sense, we study on co-even domination number of such graphs as Thorn Graphs, Thorn Path, Thorn Rod, Thorn Ring, Thorn Star, Banana Tree, Coconut Tree, Binomial Tree.

Keywords: Domination, Dominating set, Domination number, Co-even dominating set, Co-even domination number.

AMS Subject Classification: 68R10, 68M10, 05C69.

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Received: 19 March 2021; Revised: 8 June 2021; Accepted: 29 July 2021; Published: 31 August 2021.

1 Introduction

In graph theory, various measurements are used to strengthen the stability of graphically modelable communication networks. These measures are called vulnerability measures. Domination which is one of these well known measures, has a wide range of applications such as securing systems, control system design, modern science etc. Since any communication network can be modelled by a graph, we start with basic information of graphs that we use in our paper as follows.

Let G=(V,E) is a simple graph with vertex set $V(G)$ and edge set $E(G)$. For each vertex $v \in V(G)$, the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ refers to the open neighborhood of v and the set $N_G[v] = N_G(v) \cup \{v\}$ refers to the closed neighborhood of v in G (Bondy & Murty, 1976). For $S \subseteq V$ of vertices in a graph G=(V,E) is called a dominating set if every vertex $v \in V$ is either an element of S or adjacent to an element of S. The minimum cardinality of a dominating set of G is called the domination number and is denoted by $\gamma(G)$ (Haynes et al., 1998). Multiple domination, in which we insist that each vertex in $V - D$ be dominated by at least k vertices in D for a fixed positive integer k (Haynes et al., 1998). Locating domination, in which we insist that each vertex in $V - D$ has a unique set of vertices in D which dominate it (Haynes et al., 1998). Strong domination, in which we insist that each vertex v in $V - D$ be dominated by at least one vertex in D whose degree is grater than or equal to the degree of v (Haynes et al., 1998). A total dominating set, abbreviated TD-set, of a graph $G = (V, E)$ with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S (Henning et al., 2013). Ahmed A. Omran and Manar M. Shalaan presented a new definition of domination which is called co-even domination defined as follows,

Definition 1. *Let G be a graph and D is a dominating set, the set D is called co-even dominating set if, deg(v) is even number for all $v \in V$ - D. (Shalaan & Omran, 2020)*

Definition 2. *Consider G be a graph and D is a co-even dominating set, then D is called a minimal co-even dominating set if has no proper subset is a co-even dominating of G. Take us MCEDS(G) refers to all minimal co-even dominating sets of a graph G. (Shaalan and Omran, 2020)*

Definition 3. *The thorn path $P_{n,p,k}$ is obtained from the path P_n by adding p neighbors to each of its nonterminal vertices and k neighbors to each of its terminal vertices (Gutman, 1998).*

Definition 4. *A thorn rod is a graph, $P_{n,m}$, which includes a linear chain of n vertices and degree-m terminal vertices at each of the two rod ends (Gutman, 1998).*

Definition 5. *A m-thorn ring has a simple cycle as the parent, and m-2 thorns at each cycle vertex (Gutman, 1998).*

Definition 6. *The thorn star $S_{n,p,k}$ is obtained from the star S_n by adding p neighbors to the center of the star and k neighbors to its terminal vertices (Gutman, 1998).*

Definition 7. *A Banana tree $B(m,n)$ is a graph obtained by connecting one leaf of each of m copies of a n-star graph with a new single root vertex 'v' (Chen et al., 1997).*

Definition 8. *A Coconut tree $CT(m,n)$ is the graph obtained from the path P_m by appending 'n' new pendant edges at an end vertex of P_m (Sugumaran et al., 2018).*

Definition 9. *The binomial tree of order $n > 0$ with root R is the tree B_n defined as follows. If $n = 0$, $B_n = B_0 = R$, i.e., the binomial tree of order zero consists of a single node R. If $n > 0$, $B_n = R, B_0, B_1, \ldots, B_{n-1}$, i.e., the binomial tree of order $n > 0$ comprises the root R, and n binomial subtrees, $B_0, B_1, \ldots, B_{n-1}$ (Cormen et al., 1990).*

In this paper, we will examine co-even domination number of banana tree, coconut tree, binomial tree and some thorn graph classes.

2 Co-Even Domination Number of Some Path Related Graphs

Co-even domination concept was introduced by M.M. Shalaan and A.A. Omran (2020). They established bounds for some classes of graphs. They also determined exact values for paths and cycles. In this section we examine co-even domination number of binomial trees, coconut trees and banana trees and share their proofs.

Proposition 1. *Let $G = (n, m)$ be a graph and D is a co-even dominating set, then*

1. *All vertices of odd or zero degrees belong to every co-even dominating set.*
2. *$deg(v) \geqslant 2$, for all $v \in V - D$.*
3. *If G is r - regular graph then*

$$\gamma_{coe}(G) = \begin{cases} n, & \text{if } r \text{ is odd} \\ \gamma(G), & \text{if } r \text{ is even} \end{cases}$$

4. *$\gamma(G) \geqslant \gamma_{coe}(G)$ (Shaalan & Omran, 2020)*

Theorem 1. *Let B(m,n) be a banana tree where $m, n \geqslant 1$, then*

$$\gamma_{coe}(B(m,n)) = \begin{cases} m, & \text{if } m \text{ is even and } n = 1 \\ m+1, & \text{if } n = 2 \text{ or } m \text{ is odd and } n = 1 \\ m(n-1)+1, & \text{if } n \text{ is even, } n \geq 4 \\ m(n-2)+1, & \text{if } n \text{ is odd, } n \geq 3 \end{cases}$$

Proof. This proof is handled separately for 4 cases.

- **Case 1**. Let D be a co-even dominating set of $B_{m,n}$. If m is even and $n = 1$, then the degree of the vertex 'v' is even. So, we place this vertex in set $V - D$. By Proposition 1(1), all m pendant vertices of $B(m,1)$ belong to co-even dominating set D. Therefore,

$$\gamma_{coe}(B(m,1)) = m.$$

- **Case 2**. Let D be a co-even dominating set of $B_{m,n}$.

  If m is odd and $n = 1$, then the degree of the vertex 'v' is odd. So, this vertex has to be included in the set D. By Proposition 1(1), all m pendant vertices of $B(m,1)$ belong to co-even dominating set D.

  If $n = 2$, in all possible selections of the vertices, the vertex 'v' must belong to co-even dominating set D. Because if m is odd, then the degree of the vertex 'v' is odd. By the definition of co-even domination, the vertex v must be included in the set D. Otherwise, the only vertex that is not dominated in the graph is 'v', so we have to choose it.

  Therefore,

$$\gamma_{coe}(B(m,1)) = \gamma_{coe}(B(m,2)) = m + 1.$$

- **Case 3**. Let n be an even number, where $n \geq 4$.

  Let D be a co-even dominating set of $B_{m,n}$. By definition of the banana tree, the banana graph is obtained by connecting one leaf of each of m copies of a n-star graph with a new single root vertex 'v'. A n-star graph has (n-2) pendant vertices in a banana tree. Since we know that there are m copies of a n- star graph, we can calculate the total number of pendant vertices as m(n-2). By Proposition 1(1), these pendant vertices belong to co-even dominating set D.

  For $n \geq 4$, whether m is odd or even does not change the result. If n is even, then the degree of center vertices of n-star graph is odd. Since there are m center vertices in total, these vertices are included in co-even dominating set D. As it is mentioned in Case 2, also we have to choose the vertex 'v'. Therefore,

$$\gamma_{coe}(B(m,n)) = m(n-1) + 1.$$

- **Case 4**. Let n be an odd number, where $n \geq 3$.

  For $n \geq 3$, whether m is odd or even does not change the result. If n is odd, then the degree of center vertices of n-star graph is even. Since these center vertices are already dominated, we don't have to choose them. Again, as it is mentioned in Case 2, also we have to choose the vertex 'v'. Therefore,

$$\gamma_{coe}(B(m,n)) = m(n-2) + 1.$$

Therefore, from the four previous cases, the result is obtained. □

Theorem 2. *Let CT(m,n) be a coconut tree where $m, n \geqslant 1$, then*

$$\gamma_{coe}(CT(m,n)) = \begin{cases} n + \lceil \frac{m-4}{3} \rceil + 2, & \textit{if } n \textit{ is even} \\ n + \lceil \frac{m}{3} \rceil, & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. Let D be a co-even dominating set of $CT(m, n)$. By Proposition 1(1), all pendant vertices in graph belong to every co-even dominating set D, since that the degree of all these vertices is odd. So, n new pendant vertices at an end vertex of P_m belong to co-even dominating set D. Then, only the vertices of P_m remain. Since the value of m indicates the length of the P_m, whether m is odd or even does not change the result.

There are two cases, depending on whether the value of n is odd or even.

- **Case 1**. Let n be an even number and the vertex set of P_m is $\{v_1, v_2, ..., v_m\}$. Since the degree of both end vertices will be odd, these two vertices must belong to every co-even dominating set D. These vertices dominate the support vertices v_2 and $v_{m\text{-}1}$. For the remaining m-4 vertices of P_m, we have to use same methodology that was used to have a dominating set. In other words, since each vertex dominates two vertices with itself, domination is examined in groups of every 3 vertices. So, for the remaining vertices, the co-even dominating set D must consist of v_{4+3k} for $k = 0, 1, ..., \lceil \frac{m-4}{3} \rceil - 1$. Therefore,

$$\gamma_{coe}(G) = n + \lceil \tfrac{m-4}{3} \rceil + 2.$$

- **Case 2**. Let n be an even number and the vertex set of P_m is $\{v_1, v_2, ..., v_m\}$. Since the degree of v_m will be even and it is already dominated by n pendant vertices, only v_1 belong to co-even dominating set D. This vertex dominates the vertex v_2. For the remaining m vertices of P_m, we have to use same methodology that was used to have a dominating set. In other words, since each vertex dominates two vertices with itself, domination is examined in groups of every 3 vertices. So, for the remaining vertices, the co-even dominating set D must consist of v_{4+3k} for $k = 0, 1, ..., \lceil \frac{m}{3} \rceil - 2$. Therefore,

$$\gamma_{coe}(G) = n + \lceil \tfrac{m}{3} \rceil.$$

Therefore, from the two previous cases, the result is obtained. □

Theorem 3. *Let B_n be a binomial tree where $n \geqslant 1$, then*

$$\gamma_{coe}(B_n) = \begin{cases} 2(\gamma_{coe}(B_{n\text{-}1}) + 1), & \text{if } n \text{ is odd} \\ 2(\gamma_{coe}(B_{n\text{-}1}) - 1), & \text{if } n \text{ is even} \end{cases}$$

Proof. Let D be a co-even dominating set of B_n. We know that the structure of B_n includes two $B_{n\text{-}1}$ and the structure of $B_{n\text{-}1}$ includes two $B_{n\text{-}2}$.

It is easily seen that $\gamma_{coe}(B_1) = 2$ for $n = 1$ and also $\gamma_{coe}(B_2) = 2$ for $n = 2$. For $n = 3$, we know that the degree of root vertex of B_2 is even. But, since B_3 includes two B_2, since these vertices will converge on the root vertex of B_3, the degree of the root vertex of two B_2 will be odd. By Proposition 1(1), these two vertices must be included in the set D. So, $\gamma_{coe}(B_3) = 2.(\gamma_{coe}(B_2) + 1) = 6$.

Again, B_4 includes two B_3. We know that the degree of root vertex of B_3 is odd. But, since B_4 includes two B_3, since these vertices will converge on the root vertex of B_4, the degree of the root vertex of two B_3 will be even. So, that means we need to remove the root vertex of B_3 which we added in the previous step. Then, $\gamma_{coe}(B_4) = 2.(\gamma_{coe}(B_3) - 1) = 10$.

Thus, there are two cases, depending on whether the value of n is odd or even.

- **Case 1**. If n is odd, then the root vertex of B_n is also odd. By Proposition 1(1), it must belong to co-even dominating set D. Same time, the degree of this vertex will be even as it will be added in the next step. So when we calculating next step, we have to remove this root vertex first.

$$\gamma_{coe}(B_n) = 2(\gamma_{coe}(B_{n\text{-}1}) + 1).$$

- **Case 2**. If n is even, then the root vertex of B_n is even. So, we don't have to choose this vertex. Same time, the degree of this vertex will be odd as it will be added in the next step. So, when calculating the next step, we must first add one vertex to the previous step.

$$\gamma_{coe}(B_n) = 2(\gamma_{coe}(B_{n\text{-}1}) - 1).$$

When we combine all these results, we obtain the following recursive formula of co-even domination number of B_n.

$$\gamma_{coe}(B_n) = 2(\gamma_{coe}(B_{n\text{-}1}) \pm 1)$$

□

3 Co-Even Domination Number of Some Thorn Graph Classes

In this section, we give results on co-even domination number of $P_{n,p,k}$, $P_{n,m}$, $C_{n,m}$, $S_{n,p,k}$ and their proofs.

Theorem 4. *Let $P_{n,p,k}$ be a thorn path where $p, k \geqslant 1$ and $n \geqslant 2$, then*

$$\gamma_{coe}(P_{n,p,k}) = \begin{cases} 2k + (n-2)(p+1), & \text{if } p \text{ and } k \text{ are odd} \\ 2(k+1) + (n-2)p, & \text{if } p \text{ and } k \text{ are even} \\ 2k + (n-2)p + n, & \text{if } p \text{ is odd and } k \text{ is even} \\ 2k + (n-2)p, & \text{if } p \text{ is even and } k \text{ is odd} \end{cases}$$

Proof. Let D be a co-even dominating set of $P_{n,p,k}$. By Proposition 1(1), all pendant vertices belong to co-even dominating set D. There are $2k$ + (n - 2)p pendant vertices in the thorn path graph. So, these vertices must belong to co-even dominating set D. The important thing here is whether p and k are odd or even.

We distinguish four cases to obtain the co-even domination number of $P_{n,p,k}$.

- **Case 1**. If p and k are odd, then except end vertices of P_n, the degree of remaning (n - 2) vertices is odd. By Proposition 1(1), these vertices belong to co-even dominating set D. Therefore,

$$\gamma_{coe}(G) = 2k + (n-2)(p+1).$$

- **Case 2**. If p and k are even, then the degree of end vertices of P_n is odd. By Proposition 1(1), these vertices belong to co-even dominating set D. Therefore,

$$\gamma_{coe}(G) = 2(k+1) + (n-2)p.$$

- **Case 3**. If p is odd and k is even, then the degree of all vertices in graph is odd. So, all vertices must belong to co-even dominating set D. Therefore,

$$\gamma_{coe}(G) = 2k + (n-2)p + n.$$

- **Case 4**. If p is even and k is odd, then it is enough to choose only pendant vertices. Because both the degrees of all the remaining vertices in the graph is even and these vertices are already dominated by pendant vertices. Therefore,

$$\gamma_{coe}(G) = 2k + (n-2)p.$$

Therefore, from the four previous cases, the result is obtained. □

Theorem 5. *Let $P_{n,m}$ be a thorn rod where $n, m \geqslant 1$, then*

$$\gamma_{coe}(P_{n,m}) = \begin{cases} 2m + \lceil \frac{n-4}{3} \rceil, & \text{if } m \text{ is odd} \\ 2(m-1) + \lfloor \frac{n}{3} \rfloor, & \text{if } m \text{ is even} \end{cases}$$

Proof. Let D be a co-even dominating set of $P_{n,m}$. By Proposition 1(1), all pendant vertices in graph belong to every co-even dominating set D, since that the degree of all these vertices is odd. So, there are $2(m$ - 1) pendant vertices in thorn rod graph and these vertices must belong to co-even dominating set D. Therefore, the important thing is end vertices of P_n.

We distinguish two cases to obtain the co-even domination number of $P_{n,m}$.

- **Case 1**. Let m be an odd number and the vertex set of P_n is $\{v_1, v_2, ..., v_n\}$. Since the degree of both end vertices will be odd, these two vertices must belong to every co-even dominating set D. These vertices dominate the support vertices v_2 and v_{n-1}. For the remaining n-4 vertices of P_n, we have to use same methodology that was used to have a dominating set. In other words, since each vertex dominates two vertices with itself, domination is examined in groups of every 3 vertices. So, for the remaining vertices, the co-even dominating set D must consist of v_{4+3k} for $k = 0, 1, ..., \lceil \frac{n-4}{3} \rceil - 1$.

  If we arrange it as follows,

  $$= 2(m-1) + \lceil \tfrac{n-4}{3} \rceil + 2$$

  $$= 2(m-1+1) + \lceil \tfrac{n-4}{3} \rceil$$

  $$= 2m + \lceil \tfrac{n-4}{3} \rceil.$$

  Therefore,

  $$\gamma_{coe}(G) = 2m + \lceil \tfrac{n-4}{3} \rceil.$$

- **Case 2**. Let m be an even number and the vertex set of P_n is $\{v_1, v_2, ..., v_n\}$. Since the degree of v_n will be even and it is already dominated by m-1 pendant vertices, only v_1 belong to co-even dominating set D. This vertex dominates the vertex v_2. For the remaining n vertices of P_n, we have to use same methodology that was used to have a dominating set. In other words, since each vertex dominates two vertices with itself, domination is examined in groups of every 3 vertices. So, for the remaining vertices, the co-even dominating set D must consist of $v_{3(k+1)}$ for $k = 0, 1, ..., \lfloor \frac{n}{3} \rfloor - 1$. Therefore,

  $$\gamma_{coe}(G) = 2(m-1) + \lfloor \tfrac{n}{3} \rfloor.$$

Therefore, from the two previous cases, the result is obtained. □

Theorem 6. *Let $C_{n,m}$ be a m-thorn ring graph where $n, m \geqslant 1$, then*

$$\gamma_{coe}(C_{n,m}) = \begin{cases} n(m-1), & \text{if } m \text{ is odd} \\ n(m-2), & \text{if } m \text{ is even} \end{cases}$$

Proof. Let D be a co-even dominating set of $C_{n,m}$. We distinguish two cases to obtain the co-even domination number.

- **Case 1**. If m is odd, then the degree of all vertices in graph is odd. So, all vertices must be belong to co-even dominating set D. Therefore,

$$\gamma_{coe}(G) = n(m-1).$$

- **Case 2**. If m is even, by Proposition 1(1), all pendant vertices in graph belong to co-even dominating set D. There are $n.(m$ - 2) pendant vertices in the m-thorn ring graph and these vertices dominate the remaining vertices. Therefore,

$$\gamma_{coe}(G) = n(m-2).$$

Therefore, from the two previous cases, the result is obtained. □

Theorem 7. *Let $S_{n,p,k}$ be a thorn star where $n, p, k \geq 1$, then*

$$\gamma_{coe}(S_{n,p,k}) = \begin{cases} (n-1)k+p+1, & \textit{if n,p,k are odd or n,p are even and k is odd} \\ (n-1)(k+1)+p, & \textit{if n,k are even and p is odd or p,k are even and n is odd} \\ (n-1)k+p, & \textit{if n,k are odd and p is even or p,k are odd and n is even} \\ (n-1)k+p+n, & \textit{if n,p,k are even or n,p are odd and k is even} \end{cases}$$

Proof. Let D be a co-even dominating set of $S_{n,p,k}$. By Proposition 1(1), all pendant vertices in graph belong to every co-even dominating set D, since that the degree of all these vertices is odd. There are $(n$ - $1)k$ + p pendant vertices in thorn star graph. So, these vertices must belong to co-even dominating set D.

For the remaining vertices, we distinguish four cases to obtain the co-even domination number.

- **Case 1**. If n,p,k are odd or n, p are even and k is odd, then the degree of n^{th} vertex is odd. So, of the remaining vertices, only this vertex must belong to co-even dominating set D. Therefore,

$$\gamma_{coe}(G) = (n-1)k+p+1.$$

- **Case 2**. If n,k are even and p is odd or p,k are even and n is odd, then except for the n^{th} vertex, the degree of all vertices is odd. Therefore,

$$\gamma_{coe}(G) = (n-1)(k+1)+p.$$

- **Case 3**. If n,k are odd and p is even or p,k are odd and n is even, then it is enough to choose only pendant vertices. Because both the degrees of all the remaining vertices in the graph is even and these vertices are already dominated by pendant vertices. Therefore,

$$\gamma_{coe}(G) = (n-1)k+p.$$

- **Case 4**. If n,p,k are even or n,p are odd and k is even, then the degree of all vertices in thorn star graph is odd. So, all vertices must belong to co-even dominating set D. Therefore,

$$\gamma_{coe}(G) = (n-1)k+p+n.$$

Therefore, from the four previous cases, the result is obtained. □

4 Conclusion

In graph theory, domination is one of those most important concepts in the stability analysis of communication networks modelled by graphs. There are various types of domination depending on structure and properties of dominating sets. Domination has a wide range of applications.

Through, this paper, we examine a new domination parameter co-even domination number which was defined by M.M. Shalaan and A.A. Omran (2020). We find co-even domination number path related graphs such as banana tree, coconut tree, binomial tree and thorn graph classes such as thorn path, thorn rod, thorn ring, thorn star and share results.

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